Ex: 2 Find the divided differences of $f(x) = x^3 + x + 2$ for the arguments 1, 3, 6, 11.

Sol

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	4	$\frac{32-4}{3-1} = 14$	$\frac{64-14}{6-1} = 10$	$\frac{20-10}{11-1} = 1$
3	32	$= 64$	$= 20$	
6	224			
11	1344			

Ex: 3 Show that $\underset{bcd}{\Delta^3}\left(\frac{1}{a}\right) = -\frac{1}{abcd}$.

Sol

If $f(x) = \frac{1}{x}$, $f(a) = \frac{1}{a}$.

$$f(a,b) = \underset{b}{\Delta}\left(\frac{1}{a}\right) = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$$f(a,b,c) = \underset{b,c}{\Delta}\left(\frac{1}{a}\right) = \frac{f(b,c) - f(a,b)}{c-a} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c-a} = \frac{\frac{-a+c}{abc}}{c-a}$$

$$= \frac{1}{abc}\left(\frac{c-a}{c-a}\right)$$

$$= \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = \frac{1}{abcd}\left(\frac{a-d}{d-a}\right) = -\frac{1}{abcd}.$$

$$\therefore \underset{bcd}{\Delta^3} \frac{1}{a} = -\frac{1}{abcd}.$$

Properties of divided differences

The operator Δ is linear.

Pf If $f(x)$ and $g(x)$ are two functions and α, β are constants, then

$$\Delta\left[\alpha f(x) + \beta g(x)\right] = \frac{\left[\alpha f(x_1) + \beta g(x_1)\right] - \left[\alpha f(x_0) + \beta g(x_0)\right]}{x_1 - x_0}$$

$$= \alpha \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \beta \cdot \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$

$$= \alpha \Delta f(x) + \beta \Delta g(x)$$

Corollary 1: Setting $\alpha = \beta = 1$,

$$\Delta\left[f(x) + g(x)\right] = \Delta f(x) + \Delta g(x).$$

2: Setting $\beta = 0$,

$$\Delta\left[\alpha f(x)\right] = \alpha \Delta f(x).$$

Theorem

Newton's Interpolation formula for unequal intervals (Newton's divided difference formula).

Pf Let $y = f(x)$ takes values $f(x_0), f(x_1) \ldots f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$.

By definition, $f(x, x_0) = \dfrac{f(x) - f(x_0)}{x - x_0}$